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6 On the Probability Distribution
for Inventory Position in
Two Echelon Continuous Review Systems.

by

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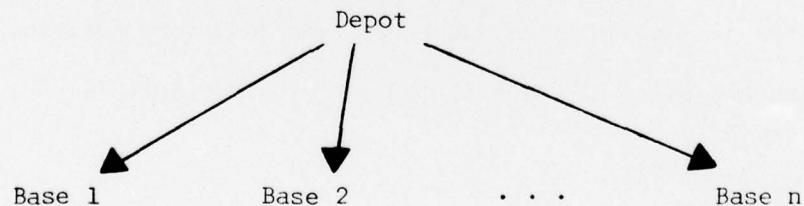
SUMMARY

In this paper ~~we~~ derive the stationary probability distribution for the inventory position at each location in a two echelon inventory system. The inventory system consists of a depot and a set of bases. All system demands are assumed to originate at a base in the second echelon. Bases are resupplied as necessary by the depot, the first echelon; the depot is resupplied by an external supplier. Each location is assumed to follow a continuous review (S,s) policy. All excess demand is assumed backordered. Furthermore, the process generating demand at each base is assumed to be a Poisson process. A simple queuing analysis is used to obtain the probability distribution for inventory position. In the case where all bases have identical arrival rates and follow the same (S,s) policy, ^{it is} ~~we~~ shown that the random variables describing the inventory position at each location are uniformly distributed and are independent.

I. INTRODUCTION

In this paper we derive the stationary probability distribution for the inventory position at each location in a two echelon inventory system. By inventory position we mean on-hand plus on-order inventory minus backorders. Once this distribution is known, the stationary probability distributions for both backorders and on-hand inventory can be found for each location in the system. These distributions can then be used to find the optimal values for the inventory policy variables.

The two echelon system we will examine consists of a depot and a set of bases. The depot is the first echelon, and the bases are the second echelon. All demands are assumed to originate at a base in the second echelon. The demands, for example, could be for a replacement part for an aircraft. The bases are resupplied as necessary by the depot; the depot, in turn, is resupplied by an external supplier. The flow in the system is displayed in Figure 1.



Inventory Flow

Figure 1

We also assume each location follows a continuous review (S,s) policy; that is, whenever the inventory position at a location falls below $s+1$, an order is immediately placed for an amount sufficient to raise that location's inventory position to S . Furthermore, if on-hand inventory at a location is insufficient to satisfy a customer order, we assume the excess demand is backordered.

The stationary probability distribution for the inventory position has been derived by many authors for single echelon systems operating under various continuous review policies [1,3,5,7,8]. This distribution has also been obtained for two echelon systems in which all bases follow continuous review $(S,S-1)$ policies [2,6].

The remainder of the paper is divided into two major parts. In the next section we show how the probability distribution for the inventory position can be obtained for each base and the depot. A simple queuing analysis is used to obtain the results. In Section III we restrict attention to the situation where all bases have the same arrival rates and follow identical (S,s) policies. In this case, the random variables describing the inventory position at each location are uniformly distributed and are independent.

II. CASE I: UNEQUAL ARRIVAL RATES AND POLICY VALUES AT THE BASES

Suppose the inventory system consists of n bases and a depot. Also, we assume in the remainder of the paper that the process generating demand at each base j is a Poisson process with rate λ_j , $j = 1, \dots, n$. We let S_j and s_j represent the values of the inventory policy variables at base j , $j = 1, \dots, n$; S_0 and s_0 denote the values for the depot policy variables; Z_j represents the inventory position for base j , $j = 1, \dots, n$; and Z_0 denotes the depot inventory position. Our objective is to derive the stationary distribution for Z_0 and Z_j , $j = 1, \dots, n$.

The distribution for Z_j has been derived by many authors. Using either a queuing argument or some results from renewal theory we can easily show that Z_j follows a discrete uniform distribution over the values s_j+1, s_j+2, \dots, S_j [1,3,5,7].

Suppose all bases follow an $(S, S-1)$ policy. Then the depot demand process is also a Poisson process with rate $\sum_{j=1}^n \lambda_j$. Consequently, Z_0 also has a uniform distribution in this case. However, if the bases follow arbitrary (S, s) policies, Z_0 may not be uniformly distributed. To study this general case we use a queuing type analysis. This approach relies on the assumption that the underlying process generating demands at the bases is a Poisson process.

We define the state of the system by an $(n+1)$ -tuple $(i; k_1, \dots, k_n)$ where i represents the depot inventory position, and k_j represents the cumulative number of demands placed at base j since that base previously requested resupply. Then

$$i \in \{s_0+1, s_0+2, \dots, S_0\} \quad \text{and}$$

$$k_j \in \{0, 1, \dots, S_j - s_j - 1\}, \quad j = 1, \dots, n.$$

Next let Ω represent the set of $(n+1)$ -tuples $(i; k_1, \dots, k_n)$. Furthermore, membership in Ω is limited to those $(n+1)$ -tuples whose first component represents a depot inventory position that is realizable given the system begins when the depot inventory position is S_0 . For example, suppose there are two bases in the system and they follow $(5,3)$ and $(7,4)$ policies, respectively. Then each order placed on the depot by base 1 is for two units and each order placed by base 2 is for three units. Consequently, if the depot inventory position is S_0 , then in the future the depot inventory position will never be S_0-1 . If, for example, the depot follows an $(8,4)$ policy, the realizable depot inventory positions are 8, 6, and 5.

Let us now write the transition probabilities for Ω . Let $Q_j = S_j - s_j$, $L = \{j: k_j = 0\}$, and $P((i; k_1, \dots, k_n); t)$ represent the probability the system is in state $(i; k_1, \dots, k_n) \in \Omega$ at time t . Since the arrival process at each base is a Poisson process, it follows that

$$P((i; k_1, \dots, k_n); t + \Delta t) = P((i; k_1, \dots, k_n); t) \cdot (1 - \sum_j \lambda_j \Delta t) + \sum_{j=1}^n P((i; k_1, \dots, k_j - 1, \dots, k_n); t) \lambda_j \Delta t$$

$$+ o(\Delta t), \quad \text{whenever } L = \emptyset;$$

$$P((i; k_1, \dots, k_n); t + \Delta t) = P((i; k_1, \dots, k_n); t) \cdot (1 - \sum_j \lambda_j \Delta t) + \sum_{j \in L} P((i + Q_j; k_1, \dots, Q_j - 1, \dots, k_n); t) \cdot \lambda_j \Delta t + \sum_{j \notin L} P((i; k_1, \dots, k_j - 1, \dots, k_n); t) \cdot \lambda_j \Delta t$$

$$+ o(\Delta t), \quad \text{whenever } L \neq \emptyset \text{ and } i < S_0; \text{ and}$$

$$P((S_0; k_1, \dots, k_n); t + \Delta t) = P((S_0; k_1, \dots, k_n); t) \cdot (1 - \sum_j \lambda_j \Delta t) + \sum_{j \in L} \sum_{r=1}^{Q_j} P((S_0 + r; k_1, \dots, Q_j - 1, \dots, k_n); t) \lambda_j \Delta t + \sum_{j \notin L} P((S_0; k_1, \dots, k_j - 1, \dots, k_n); t) \lambda_j \Delta t$$

$$+ o(\Delta t), \quad \text{whenever } L \neq \emptyset.$$

The set L refers to the state of the system at time $t + \Delta t$.

These equations describe the possible transitions in a continuous parameter Markov chain having state space Ω . Since the state space

is finite, the chain is irreducible, and the transition probability functions are homogeneous, the stationary distribution (long-run distribution) exists for the Markov chain.

The stationary distribution for the depot inventory position can be obtained using the above equations. Denote the stationary probability for state $(i; k_1, \dots, k_n) \in \Omega$ by $p(i; k_1, \dots, k_n)$. It is easy to see that the stationary probabilities satisfy

$$(1) \quad p(i; k_1, \dots, k_n) = \sum_{j=1}^n \left(\frac{\lambda_j}{\sum_m \lambda_m} \right) p(i; k_1, \dots, k_{j-1}, \dots, k_n),$$

whenever $L = \emptyset$;

$$(2) \quad p(i; k_1, \dots, k_n) = \sum_{j \in L} \left(\frac{\lambda_j}{\sum_m \lambda_m} \right) p(i+Q_j; k_1, \dots, Q_j-1, \dots, k_n) \\ + \sum_{j \notin L} \left(\frac{\lambda_j}{\sum_m \lambda_m} \right) p(i; k_1, \dots, k_{j-1}, \dots, k_n)$$

whenever $L \neq \emptyset$ and $i < S_0$;

$$(3) \quad p(S_0; k_1, \dots, k_n) = \sum_{j \in L} \sum_{r=1}^{Q_j} p(S_0+r; k_1, \dots, Q_j-1, \dots, k_n) \left(\frac{\lambda_j}{\sum_m \lambda_m} \right) \\ + \sum_{j \notin L} \left(\frac{\lambda_j}{\sum_m \lambda_m} \right) \cdot p(S_0; k_1, \dots, k_{j-1}, \dots, k_n)$$

whenever $L \neq \emptyset$; and

$$(4) \quad \sum_{i=s_0+1}^{s_0} \sum_{k_1=0}^{Q_1-1} \cdots \sum_{k_n=0}^{Q_n-1} p(i; k_1, \dots, k_n) = 1. \quad \text{The set } L \text{ in each}$$

case refers to the state on the left side of the equation.

Now let

$$(5) \quad \pi_i = \sum_{k_1=0}^{Q_1-1} \cdots \sum_{k_n=0}^{Q_n-1} p(i; k_1, \dots, k_n) \quad \text{for } i = s_0+1, \dots, S_0.$$

Then π_i measures the stationary and long-run probability that the depot inventory position is i . Thus we have shown how to determine the stationary probability distribution for depot inventory position.

As an illustration, let us find the stationary distribution for the base and depot inventory position for the following situation. Suppose the system consists of a depot and two bases. Furthermore, suppose $Q_1 = 1$, $Q_2 = 2$, $S_1 = 2$, $S_2 = 3$, $s_0 = 0$, and $S_0 = 2$. Also, let's assume the demand rate at base two is twice the demand rate at base one; that is, $2\lambda_1 = \lambda_2$. Then

$$\Omega = \{(2;0,0), (2;0,1), (1;0,0), (1;0,1)\}.$$

As we observed earlier, the inventory position at each base is uniformly distributed over the values s_j+1, \dots, S_j . Thus the probability is $1/2$ that the inventory position is 2 for base one, and the probability the inventory position at base two is 2 or 3 is $1/2$.

The probability distribution for the depot inventory position can be found using equations (1)-(5). For this case the equations are

$$p(2;0,0) + p(2;0,1) + p(1;0,0) + p(1;0,1) = 1 ,$$

$$-p(2;0,0) + \frac{\lambda_2}{\lambda_1 + \lambda_2} p(2;0,1) + \frac{\lambda_1}{\lambda_1 + \lambda_2} p(1;0,0) + \frac{\lambda_2}{\lambda_1 + \lambda_2} p(1;0,1) = 0 ,$$

$$\frac{\lambda_2}{\lambda_1 + \lambda_2} p(2;0,0) - p(2;0,1) + \frac{\lambda_1}{\lambda_1 + \lambda_2} p(1;0,1) = 0 ,$$

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} p(2;0,0) - p(1;0,0) = 0 ,$$

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} p(2;0,1) + \frac{\lambda_2}{\lambda_1 + \lambda_2} p(1;0,0) - p(1;0,1) = 0 ,$$

$$\pi_1 = p(1;0,0) + p(1;0,1), \text{ and}$$

$$\pi_2 = p(2;0,0) + p(2;0,1).$$

The solution to this set of equations is $p(2;0,0) = 3/8$,
 $p(2;0,1) = 5/16$, $p(1;0,0) = 1/8$, $p(1;0,1) = 3/16$, $\pi_1 = 5/16$, and
 $\pi_2 = 11/16$.

III. CASE II. EQUAL ARRIVAL RATES AND EQUAL POLICY VALUES AT THE BASES

In the previous section we developed a method for finding the probability distribution for the depot inventory position when the demand rates and policy values were not necessarily the same at all the bases. We will show in this section that the probability distribution for depot inventory position is a uniform distribution when both the demand rates and policy values are equal at all bases.

Let us now assume that $S_j = S$, $s_j = s$, $Q_j = S - s = Q$, and $\lambda_j = \lambda$ for all bases $j = 1, \dots, n$. Then $k_j \in \{0, \dots, Q-1\}$, $j = 1, \dots, n$. As a consequence of these assumptions we can make a number of observations:

1. Since all bases order Q units each time they place a depot order, the depot inventory position always equals one of the values $s_0 + Q, s_0 + 2Q, \dots, S_0$. In this case it also follows that S_0 and s_0 should be multiples of Q .
2. The time between events (the arrival of a demand at some base in the system) is exponentially distributed with parameter $n\lambda$. Furthermore, the probability that the next event occurs at base j is

$$\frac{\lambda}{\sum_{i=1}^n \lambda} = \frac{1}{n}, \text{ for } j = 1, \dots, n.$$

3. Since all bases have identical demand rates and the same policy values,

$$p(i; k_{1_1}, k_{2_1}, \dots, k_{n_1}) = p(i; k_{1_2}, k_{2_2}, \dots, k_{n_2})$$

whenever the number of bases having $k_{j_1} = 0$ is m_0 , $k_{j_1} = 1$ is m_1, \dots , and $k_{j_1} = Q-1$ is m_{Q-1} , and the number of bases having $k_{j_2} = 0, k_{j_2} = 1, \dots, k_{j_2} = Q-1$ is also m_0, m_1, \dots, m_{Q-1} , respectively, and $j = 1, \dots, n$.

4. Define a state $(i_1; k_{1_1}, \dots, k_{n_1})$ to be a neighbor of state $(i_2; k_{1_2}, \dots, k_{n_2})$ if state $(i_1; k_{1_1}, \dots, k_{n_1})$ can be reached when a transition occurs (a demand occurs at a base) and the state of the system is $(i_2; k_{1_2}, \dots, k_{n_2})$. Then another observation we can make is that all states have the same number of neighbors. In fact, the number of neighbors is equal to the number of bases in the system.

The assumptions and observations imply that the arrival rate into a state from its neighboring states is the same for all states; furthermore, the departure rate is the same for all states, and, of course, is equal to the arrival rate. Combining this result with the fact that the distribution of the time between arrivals to the system does not depend on the state of the system, we conclude that the proportion of time the system spends in each state is the same [4].

Corresponding to each realizable depot inventory position i there are Q^n possible states $(i; k_1, \dots, k_n)$. Since the system is equally likely to be in each state, the depot inventory position is also equally likely to be in any one of the states $i = S_0, S_0 - Q, \dots, S_0 - s_0 + Q$. Thus $p(i; k_1, \dots, k_n) = \frac{1}{Q^n \cdot M}$ and $\pi_i = \frac{1}{M}$, for $i \in \{S_0 - s_0 + Q, \dots, S_0\}$ and $k_j \in \{0, \dots, Q-1\}$, $j = 1, \dots, n$, where $M = (S_0 - s_0)/Q$.

We illustrate these observations using the following example.

Suppose the inventory system consists of a depot and two bases. Furthermore, assume $S-s = Q = 2$ and $S_0-s_0 = 4 = 2 \cdot Q$; that is, $M = 2$. Since the arrival rates are the same at each base, the probability that the next arrival to the system occurs at either base is $1/2$. Thus the following matrix is the transition matrix corresponding to the embedded Markov chain.

$$\begin{array}{c}
 (S_0;0,0) \quad (S_0;1,0) \quad (S_0;0,1) \quad (S_0;1,1) \quad (S_0-2;0,0) \quad (S_0-2;1,0) \quad (S_0-2;0,1) \quad (S_0-2;1,1) \\
 \begin{array}{l}
 (S_0;0,0) \\
 (S_0;1,0) \\
 (S_0;0,1) \\
 (S_0;1,1) \\
 (S_0-2;0,0) \\
 (S_0-2;1,0) \\
 (S_0-2;0,1) \\
 (S_0-2;1,1)
 \end{array}
 \begin{bmatrix}
 & 1/2 & 1/2 & & & & & \\
 & & & 1/2 & 1/2 & & & \\
 & & & 1/2 & 1/2 & & & \\
 & & & & & 1/2 & 1/2 & \\
 & & & & & 1/2 & 1/2 & \\
 1/2 & & & & & & & 1/2 \\
 1/2 & & & & & & & 1/2 \\
 & 1/2 & 1/2 & & & & &
 \end{bmatrix}
 \end{array}$$

Observe that the number of neighbor states for a given state and the number of states for which a given state is a neighbor is the same and equal to n . Furthermore, observe that this transition matrix is doubly stochastic. (This is also the situation for all possible values

assigned to Q , M and n .) As is well known, when the transition matrix is doubly stochastic, the probability that the system is in each state is the same. (This provides an alternative method of proving that for arbitrary values of Q , M , and n the probability the system is in each state is the same.) In this example the probability the system is in each state is $1/8$. Furthermore, the probability that the depot inventory position is either S_0 or S_0-2 is $1/2$.

We conclude this section by observing that the random variables Z_0, Z_1, \dots, Z_n are independent. Since $P(Z_j=S-k_j) = \frac{1}{Q}$, $j=1, \dots, n$, and $P(Z_0=i) = \frac{1}{M}$, $P(Z_0=i) \cdot P(Z_1=S-k_1) \cdots P(Z_n=S-k_n) = \frac{1}{(Q^n \cdot M)}$. However, as shown earlier, this is also the joint probability of the event $(Z_0=i, Z_1=S-k_1, \dots, Z_n=S-k_n)$. Hence the random variables Z_0, Z_1, \dots, Z_n are independent.

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